Exam Choice Model

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Last Edited: May 2026

1 Introduction

We're going to consider the case of a student who wants to maximize their total grade across the n classes (maybe n = 3,4,5) they're taking this semester.

For every class, we will define hours worked in that class as h_i, where i $\in$ {1,2,3,4,...} indexes a class number. We will define h_l as the hours spent on leisure. Finally, we define the grade function for each class, $g_i(\cdot)$, which is a function of hours worked: $g_i(h_i)$.

To start, we're going to make a number of of assumptions which we will later relax. We will list the assumptions in order of ease of justification:

1. The student has a total amount of time, $\tilde{T}$, which they can apply to either studying for a class or devote to leisure (which includes essentials like eating and sleeping).

2. In every class, the professor restricts each grade to be between 0 and 100. Or, for every amount of hours worked in the class $h_i \in \mathbb{R}$, $0 \leq g_i(h_i) \leq 100$

3. Hours worked in one class have no impact on grade in another class.

4. For every class, class grade $g_i(\cdot)$ is a continuous and strictly monotonically increasing function of hours worked h_i.

5. Furthermore, class grade has the following property: $g_i''(h_i) < 0$. (Note that assumption 4 specifies that $g_i'(h_i) > 0$).

6. The student has already solved for utility maximizing leisure time, which I will call h_l^*.

Let's consider the base case with all assumptions above. To start, note that by assumptions 1 and 6, the student can only spend an amount of time T, $T := \tilde{T} - h_l^*$, on hours spent studying. This means that the student is *constrained* by T: a student with n classes must set

$$h_1 + h_2 + ... + h_n = T \tag{1}$$

$$\Longleftrightarrow \tag{2}$$

$$\sum_{i=1}^{n} h_i = T \tag{3}$$

Using the Lagrangian, the student can then set up the maximization problem:

$$max_{h_1,h_2,...,h_n} g_1(h_1) + g_2(h_2) + ... + g_n(h_n) + \lambda(T - h_1 - h_2 - ... - h_n) \tag{4}$$

$$\Longleftrightarrow$$

$$max_{h_1,h_2,...,h_n} \sum_{i=1}^{n} g_i(h_i) + \lambda(T - \sum_{i=1}^{n} h_i) \tag{5}$$

Taking first order conditions (FOC) and second order conditions (SOC), we get a set of n FOC's and n SOC's:

FOC's:

$$g_i'(h_i) = \lambda, i \in \{1, 2, 3, ..., n\} \tag{6}$$

$$\Longleftrightarrow$$

$$g_1'(h_1) = g_2'(h_2) = ... = g_n'(h_n) \tag{7}$$

SOC's:

$$g_i''(h_i) < 0, i \in \{1, 2, 3, ..., n\} \tag{8}$$

by assumption 5. By the FOC's, if the student knows (or can approximate) each function $g_i(\cdot)$, they can solve for each total score maximizing hours worked, h_i^*. The solution steps work as follows:

1. Choose some h_i. For convenience, we'll choose h_1.
2. Recall that $g_1'(h_1) = g_j'(h_j)$ for each other class h_j. Solve for h_j in terms of h_1, or $h_j(h_1)$.
3. Recall that by equation 1, T $= h_1 + h_2(h_1) + h_3(h_1) + ... + h_n(h_1)$
4. Solve for h_1 in terms of T. This is h_1^*
5. Solve for each $h_j(h_1)$ in terms of h_1^*. These are h_j^*.

1.1 Worked Example

Suppose the student is taking 3 classes and has budgeted T = 1,000 hours to study for these classes. The grade functions for these classes are as follows:

- $g_1(h_1) = 3\sqrt{h_1}$
- $g_2(h_2) = \begin{cases} 0 & \text{if } h_2 < 1, \\ 15 * log(h_2) & \text{if } 1 \leq h_2 \leq 786, \\ 100 & \text{otherwise} \end{cases}$
- $g_3(h_3) = 10 * h_3^{1/3}$

The student assumes that $g_2(h_2)$ will be within the bounds of (0,100). In other words, they assume that $g_2(h_2) = 15 * log(h_2)$, and they will confirm that this is true later. The student then sets up the Lagrangian:

$$max_{h_1,h_2,h_3} \sum_{i=1}^{3} g_i(h_i) + \lambda(T - h_1 - h_2 - h_3) \tag{9}$$

And by the FOC's:

$$\frac{3}{2\sqrt{h_1}} = \frac{15}{h_2} = \frac{10}{3h_3^{2/3}} \tag{10}$$

Choosing h_1 as their variable of interest, the student then solves for $h_2(h_1)$:

$$3h_2 = 30\sqrt{h_1} \tag{11}$$

$$h_2 = 10\sqrt{h_1} \tag{12}$$

and solves for $h_3(h_1)$:

$$\frac{3}{2\sqrt{h_1}} = \frac{10}{3h_3^{2/3}} \tag{13}$$

$$\iff$$

$$20\sqrt{h_1} = 9h_3^{2/3} \tag{14}$$

$$\iff$$

$$\frac{20}{9}\sqrt{h_1} = h_3^{2/3} \tag{15}$$

$$\iff$$

$$(\frac{20}{9}\sqrt{h_1})^{3/2} = h_3 \tag{16}$$

Plugging back into the constraint, the student finds that

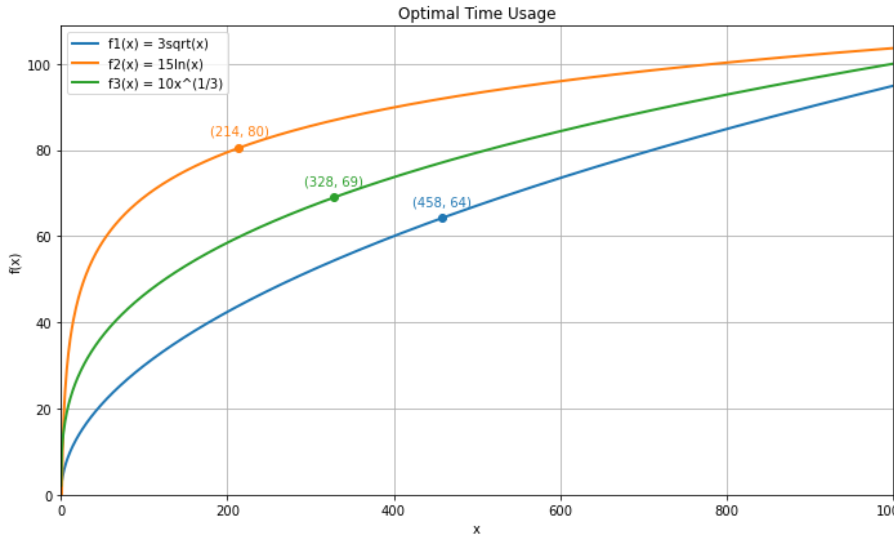

Figure 1: Enter Caption

$$T = 1000 = h_1 + 10\sqrt{h_1} + (\frac{20}{9}\sqrt{h_1})^{3/2} \tag{17}$$

Solving, the student gets that $h_1^* \approx 458$. Plugging this into equations 12 and 16, the student finds that $h_2^* \approx 214$ and $h_3^* \approx 328$. Summing, we get 458 + 214 + 328 = 1000. Note that the student's assumption that $g_2(h_2) \in (0, 100)$ held since $g_2(214) \approx 80$.

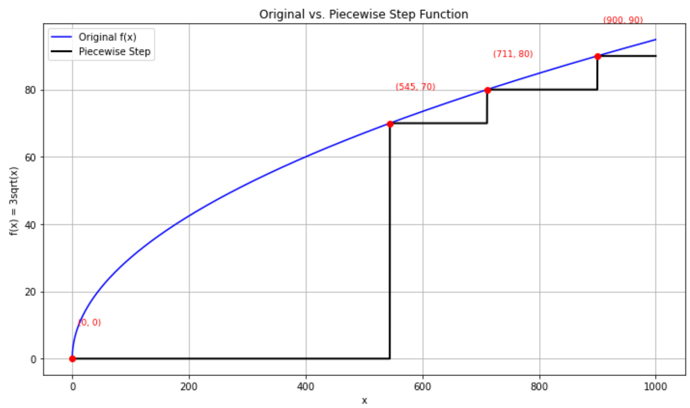

Figure 2: Enter Caption

Justification list:

1. Suppose the student has unlimited time. Then they simply devote the amount of time needed in each class to achieve a 100, and dedicate the remaining (infinite) time to leisure.

2. The numbers start at 1 with each use of the `enumerate` environment.

3. Another entry in the list

2 Piecewise Grades

Many school systems utilize grade thresholds that are piecewise in nature. Consider the US grading system. Typically, in university, final scores can only take the form of A,B,C, or fail, with many additionally allowing +/- scores (A-, B+, etc.). These grades are met by certain thresholds, most often 90,80, and 70 for A,B,C. For these grade systems, a student who is budgeting their time cannot in fact score a 75 as their final grade, they score a 70. How can they respond to this?

Consider a student with n classes in a university system with grade thresholds g_m or $\{g_1, g_2, \ldots, g_m\}$ such that $g_1 < g_2 < \ldots < g_m$. For example, consider a student with the set of thresholds 0,70,80,90 and the underlying grade function g(h) = $3\sqrt{h}$. Their final grade function looks like the green line in figure 2.

2.1 Lemma 1

If the underlying grade function g(h) is strictly monotonically increasing (as in Assumption 4), then there exists a minimum number of hours worked to reach each threshold g_k.

Proof: Note that by Assumption 4, g(h) is bijective, so there exists one value h such that $g^{-1}(h) = g_k$. This value h is the minimum.

2.2 Lemma 2

If the underlying grade function g(h) is strictly monotonically increasing (as in Assumption 4), then a student who spends a number hours h on a class, such that $g_k = g(h) < g_{k+1}$, is no worse off by spending $h^* < h$ hours on the class such that $g^{-1}(h^*) = g_k$.

Proof: The proof is trivial since $g_k = g(h^*) = g(h)$.